Mechanism Design with Monetary Transfers

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Outline of the Talk

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1. The Setup
   - Unrestricted Preferences
   - Restricted Preferences
2. Mechanisms in Quasi-linear Domain
   - Structure of a Mechanism
   - Some Definitions
3. Results
   - Groves Class of Mechanisms
   - What Other Mechanisms are Incentive Compatible
   - Revenue Equivalence
   - Uniqueness of Groves for Efficiency
   - Budget Balance
   - Bayesian Incentive Compatibility
4. Summary

Outline of the Talk

The Gibbard-Satterthwaite Setting

- Voters can have arbitrary *strict ordinal* preferences over the set of alternatives
- Set of alternatives $X = \{a, b, c, d\}$

Voter 1	Voter 2	Voter 3	Voter 4
a	d	c	d
b	b	b	b
c	a	a	c
d	c	d	a

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Theorem (Gibbard (1973), Satterthwaite (1975))

If $|X| \geq 3$, an onto social choice function *is strategyproof if and only if it is dictatorial.*

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- Each agent i has a valuation function $v_i : A \to \mathbb{R}$ belonging to the set V_i
- Agents' utilities are given by

$$u_i(x) = u_i(a, p) = v_i(a) - p_i$$

Example: Public Good

Alternatives →		
Alice	10	80
Bob	100	30
Carol	40	50

Photo courtesy: wikimedia.org and nimsuniversity.org

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- Valuations: $v_A(F) = 10, v_A(L) = 80$
- Social planner takes the decision of building F or L
- Can *tax* people differently depending on their preferences

Quasi-linear preferences

Example: Resource Allocation

Commodities →	IBMSmartCloud	hp Cloud	CISCO
Alice	0.2	0.8	0.5
Bob	0.3	0.1	0.2
Carol	0.5	0.1	0.3

Photo courtesy: individual organizations

Example: Resource Allocation

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- Set of allocations $A = \{x \in [0,1]^{n \times m} : \sum_{j=1}^{m} x_{i,j} = 1\}$
- Items are divisible among the agents
- Agents' valuations reflect their preferences over different allocations
- They are charged monetary transfers for every allocation

Quasi-linear preferences

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Selfish valuations

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  - Consider two alternatives $x_1 = (a, p_1)$ and $x_2 = (a, p_2)$, where $p_1 < p_2$
  - For all agents, $x_1 \succ x_2$ for any valuation profile
  - There is no preference profile where $x_2 \succ x_1$

An Example of a Truthful Mechanism

			
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  - pay every agent i an amount $\sum_{j \neq i} v_j(a^*)$

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- Consider the mechanism:
  - pick the alternative a^* that maximizes the sum of the valuations (with arbitrary tie-breaking rule)
  - pay every agent i an amount $\sum_{j \neq i} v_j(a^*)$
- The mechanism is truthful, even though not a dictatorship

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Structure of a Mechanism

- Set of agents $N = \{1, \dots, n\}$
- Set of allocations A, finite (for this tutorial)
- Valuation of agent i, $v_i : A \to \mathbb{R}$, the set of valuations is denoted by V_i

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- Valuation of agent i, $v_i : A \to \mathbb{R}$, the set of valuations is denoted by V_i
- A mechanism in quasi-linear (QL) domain is a pair of functions:
  - allocation function, $a : \prod_j V_j \to A$
  - payment function, $p_i : \prod_j V_j \to \mathbb{R}$, for all $i \in N$
- Agent i's payoff is given by:

$$v_i(a(v)) - p_i(v)$$

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- Only *direct revelation mechanisms* (DRM) **(this talk)**

Outline of the Talk

Social Choice Function

Definition (Social Choice Function)

A *social choice function* (SCF) f is a mapping from the set of valuation profiles to the set of allocations, i.e., $f : V \to A$, where $V = \prod_j V_j$.

- Note that the outcome is only the allocations

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- Note that the outcome is only the allocations
- In QL domain:
  A mechanism $M = (a, p)$ *implements* a SCF f if:
  - $a(v) = f(v), \forall v \in V$ and,
  - for every agent $i \in N$, reporting v_i truthfully is an *equilibrium*
- Even though the SCF is concerned with only allocations, payments can also be characterized by *revenue equivalence* (defined later)

Incentive Compatibility

Definition (Dominant Strategy Incentive Compatibility (DSIC))

A mechanism $(f, p_1, \ldots, p_n)$ is *dominant strategy incentive compatible* if for all $i \in N$ and for all $v_{-i} \in V_{-i} := \prod_{j \neq i} V_j$,

$$v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v_i', v_{-i})) - p_i(v_i', v_{-i}), \ \forall v_i, v_i' \in V_i.$$

In this case, payments $p_i, i \in N$ implement f in dominant strategies

Incentive Compatibility (Contd.)

- In a Bayesian game, the valuations v are generated through a prior P
- Each agent i knows her own realized valuation v_i and P
- Her belief on the valuations of other agents v_{-i} is given by $P(v_{-i}|v_i)$ derived by Baye's rule

Incentive Compatibility (Contd.)

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- Each agent i knows her own realized valuation v_i and P
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Definition (Bayesian Incentive Compatibility (BIC))

A mechanism $(f, p_1, \dots, p_n)$ is *Bayesian incentive compatible* for a prior P if for all $i \in N$,

$$\mathbb{E}_{v_{-i}|v_i}[v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i})] \geq \mathbb{E}_{v_{-i}|v_i}[v_i(f(v_i', v_{-i})) - p_i(v_i', v_{-i})] \quad \forall v_i, v_i' \in V_i.$$

In this case, payments $p_i, i \in N$ implement f in a Bayesian Nash equilibrium

Observations on IC

- A DSIC mechanism is always BIC

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For a DSIC mechanism $(f, p_1, \ldots, p_n)$, let valuations of agents other than i is fixed at v_{-i}

- If v_i, v_i' be such that $f(v_i, v_{-i}) = f(v_i', v_{-i})$, then $p_i(v_i, v_{-i}) = p_i(v_i', v_{-i})$

Observations on IC

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- If v_i, v_i' be such that $f(v_i, v_{-i}) = f(v_i', v_{-i})$, then $p_i(v_i, v_{-i}) = p_i(v_i', v_{-i})$
- Consider another payment $q_i(v_i, v_{-i}) = p_i(v_i, v_{-i}) + h_i(v_{-i})$,

$$v_i(f(v_i, v_{-i})) - q_i(v_i, v_{-i}) \geq v_i(f(v_i', v_{-i})) - q_i(v_i', v_{-i}), \ \forall v_i, v_i' \in V_i.$$

Efficiency

Definition (Efficiency)

An SCF f is *efficient* if for all $v \in V$,

$$f(v) \in \operatorname*{argmax}_{a \in A} \sum_{i \in N} v_i(a).$$

An efficient SCF ensures that the 'social welfare' is maximized

Revenue Equivalence

- This property characterizes the payment functions

Definition (Revenue Equivalence)

An SCF f satisfies *revenue equivalence* if for any two payment rules p and p' that implement f, there exist functions $\alpha_i : V_{-i} \to \mathbb{R}$, such that,

$$p_i(v_i, v_{-i}) = p'_i(v_i, v_{-i}) + \alpha_i(v_{-i}), \ \forall v_i \in V_i, \forall v_{-i} \in V_{-i}, \forall i \in N.$$

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- Saw an example of a payment of agent i being different by a factor not dependent on i's valuation

- This property says more: pick *any* two payments that implement f - they must be different by a similar factor

Budget Balance

Definition (Budget Balance)

A set of payments $p_i : V \to \mathbb{R}, i \in N$ is budget balanced if,

$$\sum_{i \in N} p_i(v) = 0, \forall v \in V.$$

- This property ensures that the mechanism does not produce any monetary surplus
- Hard to satisfy with incentive compatibility

Outline of the Talk

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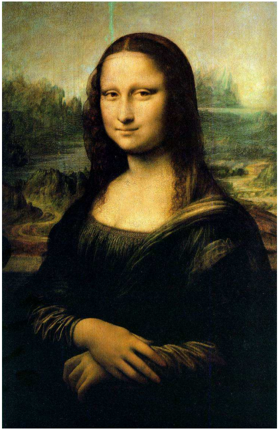
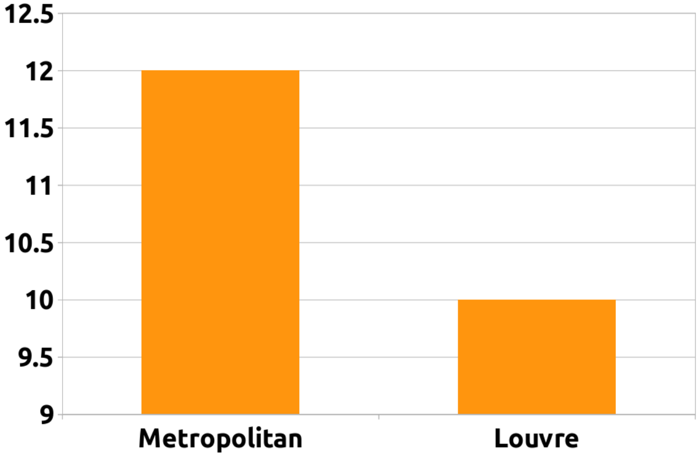
Single Indivisible Item Auction



Buyer 1
Metropolitan Museum of Arts

Buyer 2
Louvre

Second Price Auction



- Metropolitan wins, but pays second highest bid
- The mechanism is DSIC (why?)

Groves Class of Mechanisms

- Allocation rule is efficient:

$$a^*(v) \in \operatorname*{argmax}_{a \in A} \sum_{i \in N} v_i(a)$$

- Payment rule is given by:

$$p_i^*(v_i, v_{-i}) = h_i(v_{-i}) - \sum_{j \in N \setminus \{i\}} v_j(a^*(v)),$$

where $h_i : V_{-i} \to \mathbb{R}$ is any arbitrary function that does not depend on v_i

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Claim

Groves class of mechanisms are DSIC

Incentive Compatibility of Groves

- Utility of agent i according to Groves class of mechanisms:

$$u_i^{(a^*,p^*)}(v_i, v_{-i})$$
$$= v_i(a^*(v_i, v_{-i})) - p_i^*(v_i, v_{-i})$$

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& = v_i(a^*(v_i, v_{-i})) - h_i(v_{-i}) + \sum_{j \in N \setminus \{i\}} v_j(a^*(v_i, v_{-i}))
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& \geq \sum_{j \in N} v_j(a^*(v_i', v_{-i})) - h_i(v_{-i}) \text{ (by definition of } a^*)
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Pivot Mechanism

- A special case of Groves class when the payment is given by:

$$h_i(v_{-i}) = \sum_{j \in N \setminus \{i\}} v_j(a^*_{-i}(v_{-i})),$$

where the allocation $a^*_{-i}(v_{-i})$ is given by:

$$a^*_{-i}(v_{-i}) \in \underset{a \in A}{\operatorname{argmax}} \sum_{j \in N \setminus \{i\}} v_j(a)$$

- The allocation a^*_{-i} maximizes the sum of valuations in the absence of agent i
- The function h_i is the maximum value of this sum
- The payment is therefore:

$$p_i(v_i, v_{-i}) = \max_{a \in A} \sum_{j \in N \setminus \{i\}} v_j(a) - \sum_{j \in N \setminus \{i\}} v_j(a^*(v))$$

Interpretations of the Pivot Mechanism

$$p_i(v_i, v_{-i}) = \max_{a \in A} \sum_{j \in N \setminus \{i\}} v_j(a) - \sum_{j \in N \setminus \{i\}} v_j(a^*(v))$$

Two Interpretations:

1. Externality:
   - $\max_{a \in A} \sum_{j \in N \setminus \{i\}} v_j(a)$ is what the agents $N \setminus \{i\}$ can achieve
   - $\sum_{j \in N \setminus \{i\}} v_j(a^*(v))$ is what they achieve under the efficient rule
   - The mechanism asks agent i to pay the difference

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   - The mechanism asks agent i to pay the difference
2. Marginal contribution:
   - Net utility of agent i in pivot mechanism:

     $$u_i(v_i, v_{-i}) = \sum_{j \in N} v_j(a^*(v_i, v_{-i})) - \max_{a \in A} \sum_{j \in N \setminus \{i\}} v_j(a)$$

     i.e., the difference in sum valuation in presence of agent i and in her absence
   - Net utility is agent i's *marginal contribution*

What is Pivotal about it?

Alternatives →		
Alice	10	70
Bob	100	10
Carol	10	50

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- Outcome: `L`

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Bob	100	10
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- Outcome: `L`
- Alice pays $(100 + 10) - (10 + 50) = 50$

What is Pivotal about it?

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- Outcome: L
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- Carol pays $(10 + 100) - (70 + 10) = 30$

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- Alice pays $(100 + 10) - (10 + 50) = 50$
- Bob pays $(70 + 50) - (70 + 50) = 0$
- Carol pays $(10 + 100) - (70 + 10) = 30$

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Outline of the Talk

Affine Maximizers

An important class of SCFs is that of affine maximizers

Definition (Affine Maximizer)

An SCF $f : V \to A$ is an *affine maximizer* if there exists $w_i \geq 0, i \in N$, not all zero, and a function $\kappa : A \to \mathbb{R}$ such that,

$$f(v) \in \operatorname*{argmax}_{a \in A} \left(\sum_{i \in N} w_i v_i(a) + \kappa(a) \right).$$

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Special cases:

- $w_i = 1, \forall\, i$ and $\kappa \equiv 0$: **efficient** SCF

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Special cases:

- $w_i = 1, \forall\ i$ and $\kappa \equiv 0$: **efficient** SCF
- $w_d = 1$, for some d, $w_i = 0, \forall\ i \neq d$ and $\kappa \equiv 0$: **dictatorial** SCF

Affine Maximizers (Contd.)

- An affine maximizer f satisfies *independence of irrelevant agents* (IIA) if for every i with $w_i = 0$ and for every $v_{-i} \in V_{-i}$,

$$f(v_i, v_{-i}) = f(v_i', v_{-i}), \forall v_i, v_i' \in V_i$$

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- This is a consistency condition for tie-breaking
- Every affine maximizer satisfying IIA is implementable
- In particular, payments are of the following form: for all $i \in N$

$$p_i(v_i, v_{-i}) = \begin{cases} \frac{1}{w_i} \left(\sum_{j \neq i} w_j v_j(f(v)) + \kappa(f(v)) + h_i(v_{-i}) \right), & w_i > 0 \\ 0 & w_i = 0 \end{cases}$$

f is an affine maximizer

Roberts' Theorem

Theorem (Roberts 1979)

Let the allocation space A be finite with $|A| \geq 3$. If the space of valuations V is unrestricted, *then an onto and dominant strategy implementable SCF $f : V \rightarrow A$ is an affine maximizer.*

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Understanding Roberts' Theorem:

- Groves' or pivotal mechanisms are implementable, but this result is giving a necessary condition for implementability
- Moreover, it provides a functional form characterization of the DSIC mechanisms (as opposed to Myerson's monotonicity characterization)
- If payments are enforced to be zero for every valuation profile v, then the only implementable mechanism is dictatorial - GS theorem is a corollary of this result

Some Observations and Implications

- If an SCF f is implementable in a valuation space V, it is implementable in every valuation space $V' \subseteq V$ - same payments implement them and the number of incentive compatibility constraints reduce

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- Efficient SCF is implementable in any valuation space
- Unrestricted valuation space is crucial for Roberts' theorem - some recent results show that the affine maximizer characterization is true even for certain restricted valuation spaces
- Characterization of implementability in restricted domains is an active research area

[A proof by pictures]

Outline of the Talk

Revenue Equivalence

- If p and p' implement f in dominant strategies, then

$$p_i(v) = p'_i(v) + \alpha_i(v_{-i}), \forall v \in V$$

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Theorem (Chung and Olszewski (2007))

Suppose the type space $T \subseteq \mathbb{R}^n$ is a connected set, A is finite and the valuations are continuous in type. If an SCF is implementable in dominant strategies, then it satisfies revenue equivalence.

Outline of the Talk

Green-Laffont-Holmström Characterization

- An efficient SCF f chooses an alternative in $\operatorname{argmax}_{a \in A} \sum_{j \in N} v_j(a)$

Green-Laffont-Holmström Characterization

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Theorem (Green and Laffont (1979), Holmström (1979))

If the valuation space is convex and smoothly connected, every efficient and DSIC mechanism is a Groves mechanism.

- Shows uniqueness of Groves class in the space of efficient, DSIC mechanisms

[A PROOF OUTLINE]

Outline of the Talk

Green-Laffont Impossibility

Theorem (Green and Laffont (1979))

No Groves mechanism is budget balanced *(BB), i.e.,*
$\nexists p_i^{\text{Groves}} s.t. \sum_{i \in N} p_i^{\text{Groves}}(v) = 0, \forall v \in V.$

- This leads to the following corollary

Corollary

If the valuation space is convex and smoothly connected, no efficient *mechanism can be both DSIC and BB.*

Outline of the Talk

AGV Mechanism

- If the equilibrium condition is relaxed to BIC, we have a positive result
- Payment is defined via a function $\delta_i, i \in N$:

$$\delta_i(v_i) = \mathbb{E}_{v_{-i}|v_i} \left(\sum_{j \in N \setminus \{i\}} v_j(a^*(v)) \right),$$

  where a^* is an efficient allocation
- Payment is:

$$p_i^{\text{AGV}}(v) = \sum_{j \in N \setminus \{i\}} \delta_j(v_j) - \delta_i(v_i)$$

Theorem (d'Aspremont and Gerard-Varet (1979), Arrow (1979))

The AGV mechanism is BIC, efficient, and budget-balanced

Outline of the Talk

Summary

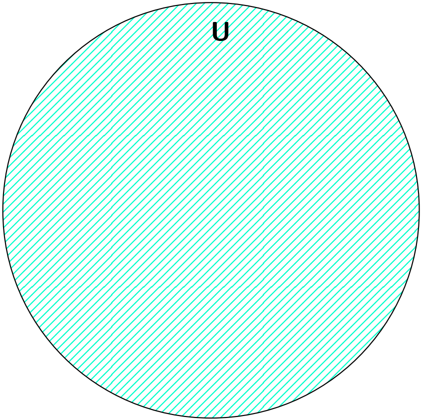
DSIC mechanisms

Valuation / Type Space

Mechanism Space

Summary

DSIC mechanisms



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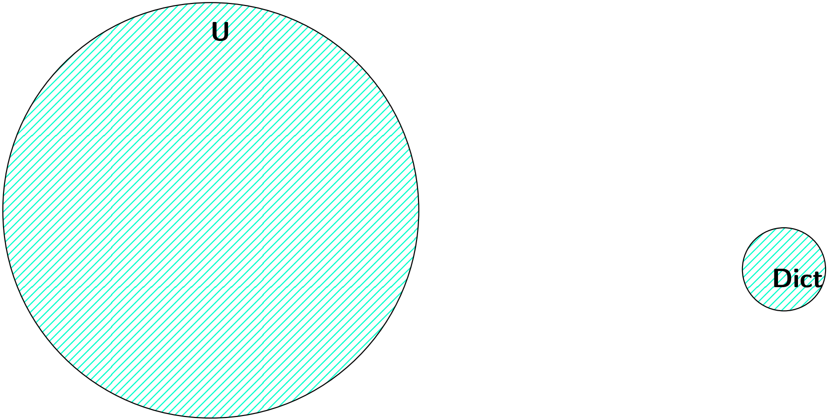

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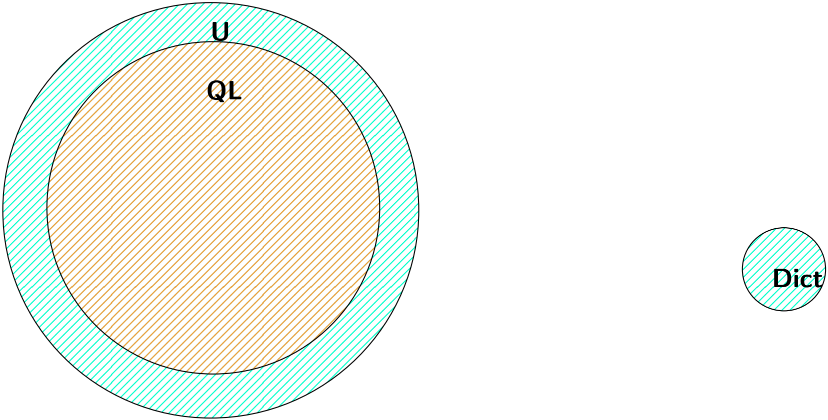

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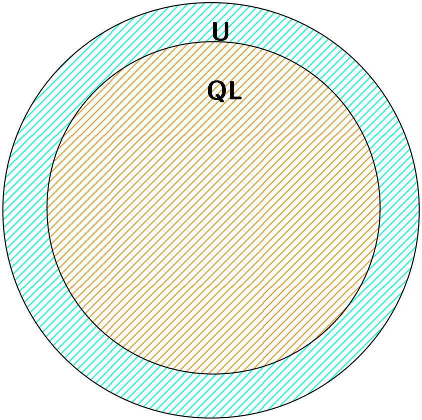

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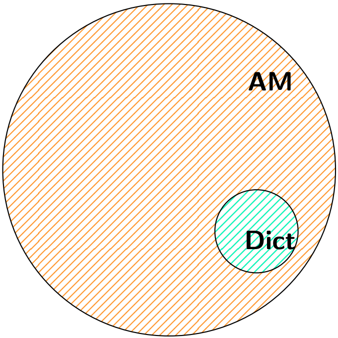

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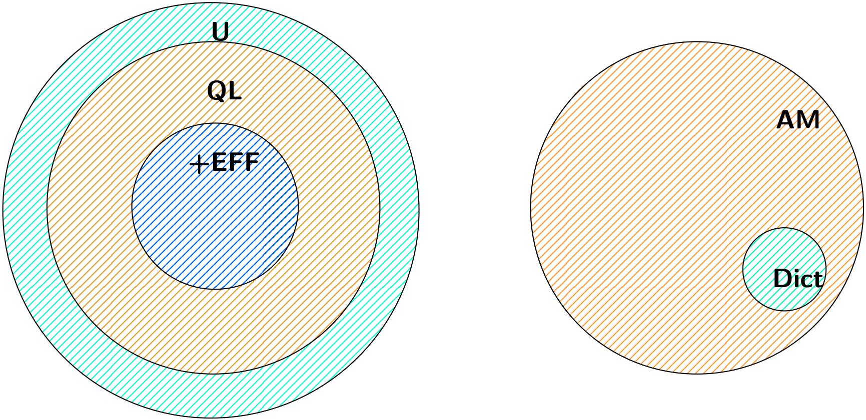

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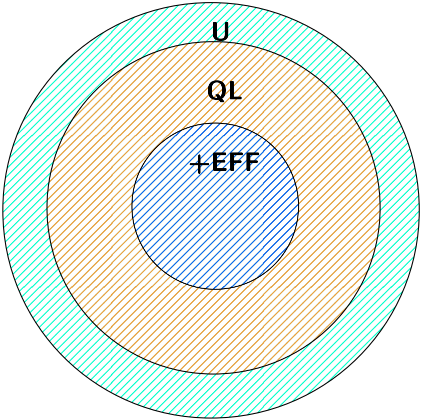

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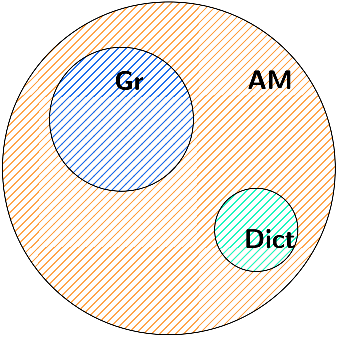

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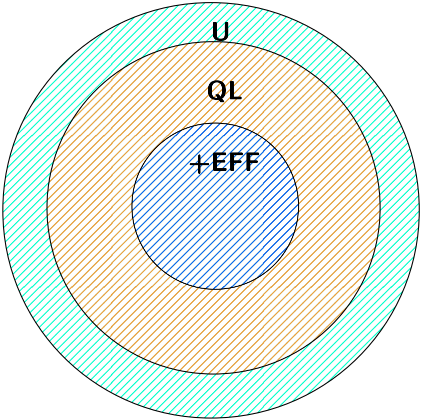

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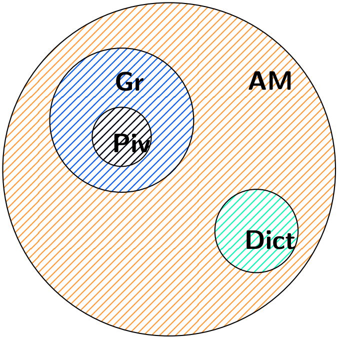

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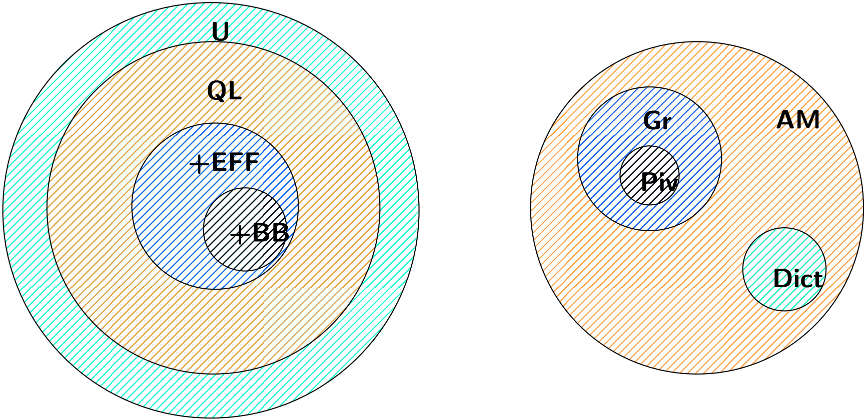

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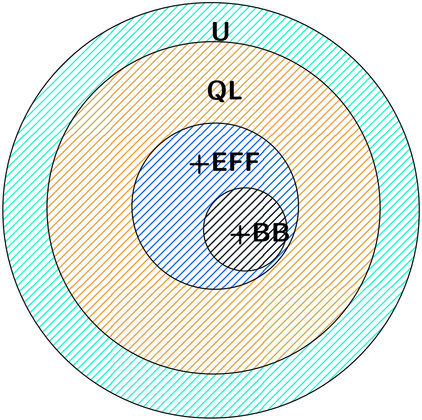

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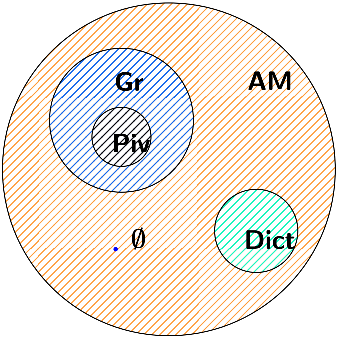

Mechanism Space

Thank you!

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http://www.isid.ac.in/∼swaprava

Value Difference Set

Q: What does affine maximizer mean?

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A: If $f(v) = y$ then

$$\begin{aligned} & w^\top v(y) + \kappa(y) \geq w^\top v(z) + \kappa(z), \forall z \in A \setminus \{y\} \\ \Rightarrow\ & w^\top (v(y) - v(z)) \geq \kappa(z) - \kappa(y), \forall z \in A \setminus \{y\} \end{aligned}$$

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- Define the *value difference set* for any pair of *distinct* alternatives $y, z \in A$.

$$P(y,z) = \{\alpha \in \mathbb{R}^n : \exists\ v \in V \text{ s.t. } v(y) - v(z) = \alpha \text{ and } f(v) = y\}.$$

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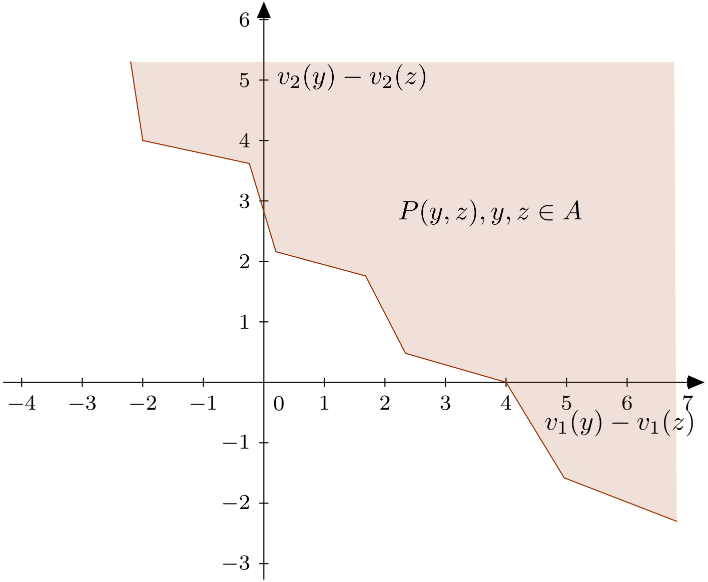
Claim

If $\alpha \in P(y, z)$, and $\delta > \mathbf{0} \in \mathbb{R}^n$, then $\alpha + \delta \in P(y, z)$, for all distinct $y, z \in A$.

Graphical Illustration for Two Players



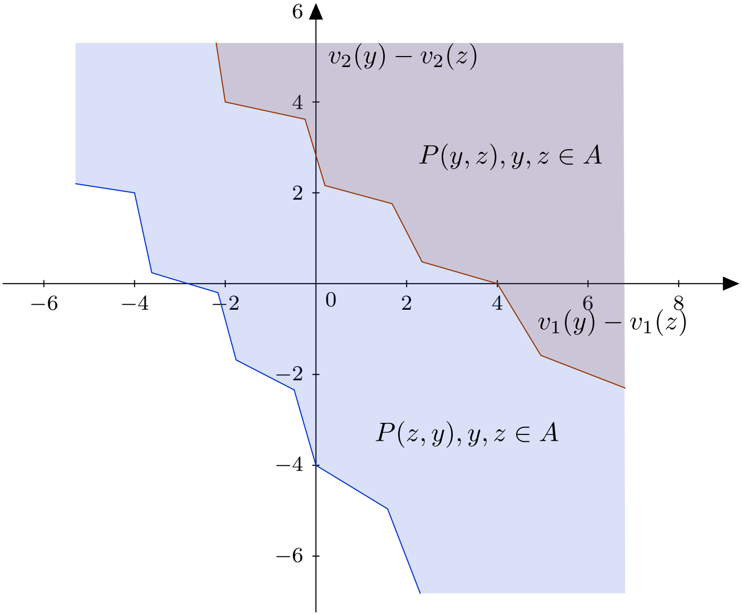
Complementary Structures of $P(y, z)$ and $P(z, y)$

Claim

For every $\alpha, \epsilon \in \mathbb{R}^n,\ \epsilon > \mathbf{0}$, *and for all* $y, z \in A$,

(a) $\alpha - \epsilon \in P(y, z) \quad \Rightarrow \quad -\alpha \notin P(z, y)$.

(b) $\alpha \notin P(y, z) \quad \Rightarrow \quad -\alpha \in P(z, y)$.



6
$v_2(y) - v_2(z)$
4
$P(y,z), y,z \in A$
2
−6
−4
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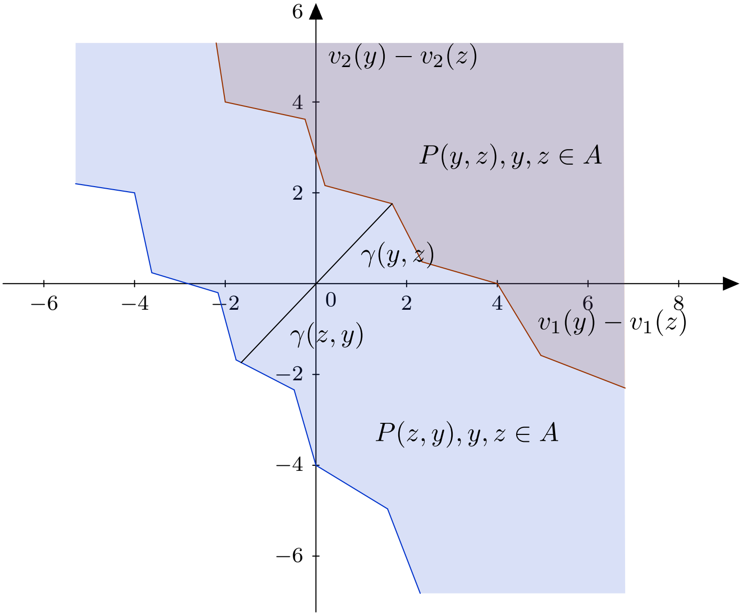
6
8
$v_1(y) - v_1(z)$
−2
$P(z,y), y,z \in A$
−4
−6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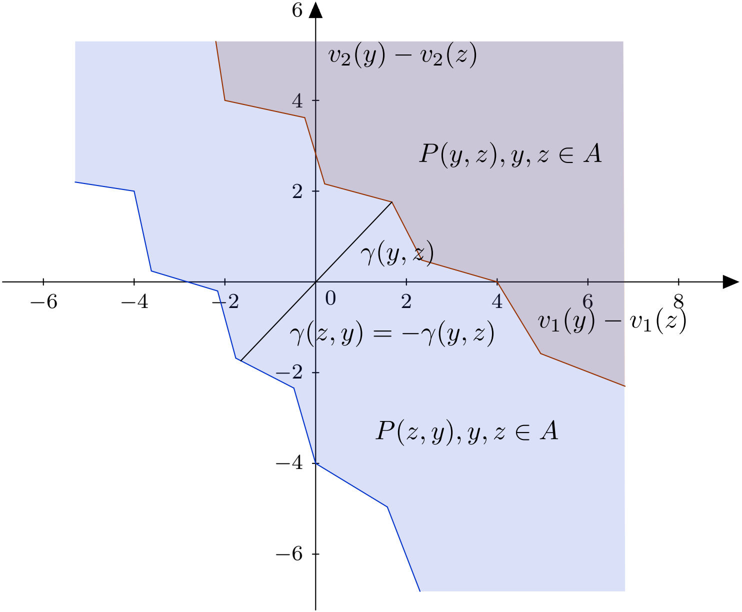
$v_1(y) - v_1(z)$
$P(y, z), y, z \in A$
$P(z, y), y, z \in A$
$\gamma(y, z)$
$\gamma(z, y)$



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0
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$v_2(y) - v_2(z)$
$P(y,z), y,z \in A$
$\gamma(y,z)$
$v_1(y) - v_1(z)$
$\gamma(z,y) = -\gamma(y,z)$
$P(z,y), y,z \in A$

Independence of $\mathring{C}$ from the Alternatives in A

- Define the translated set $C(y,z) = P(y,z) - \gamma(y,z)\mathbf{1}$
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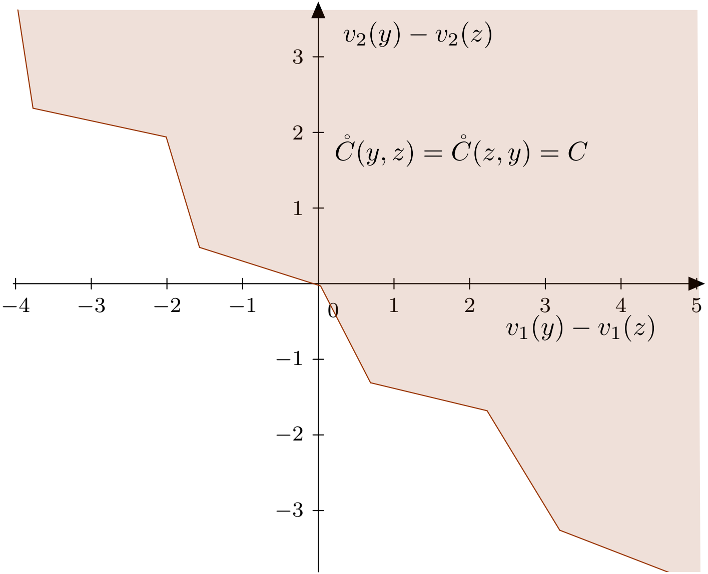
Remark: Note that this result, in particular, includes the cases, $\mathring{C}(y,z) = \mathring{C}(l,z) = \mathring{C}(l,y) = \mathring{C}(z,y)$. Therefore, the claim holds even without y,z,w,l being all distinct.


$v_2(y) - v_2(z)$
$\mathring{C}(y,z) = \mathring{C}(z,y) = C$
$v_1(y) - v_1(z)$
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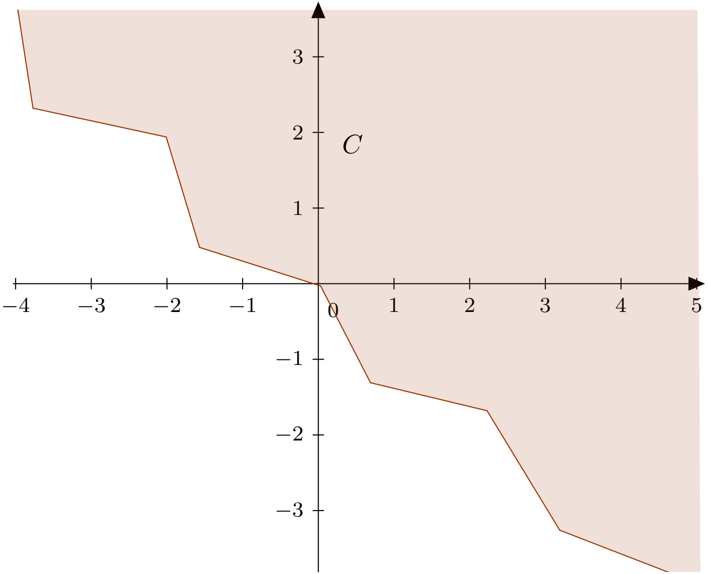
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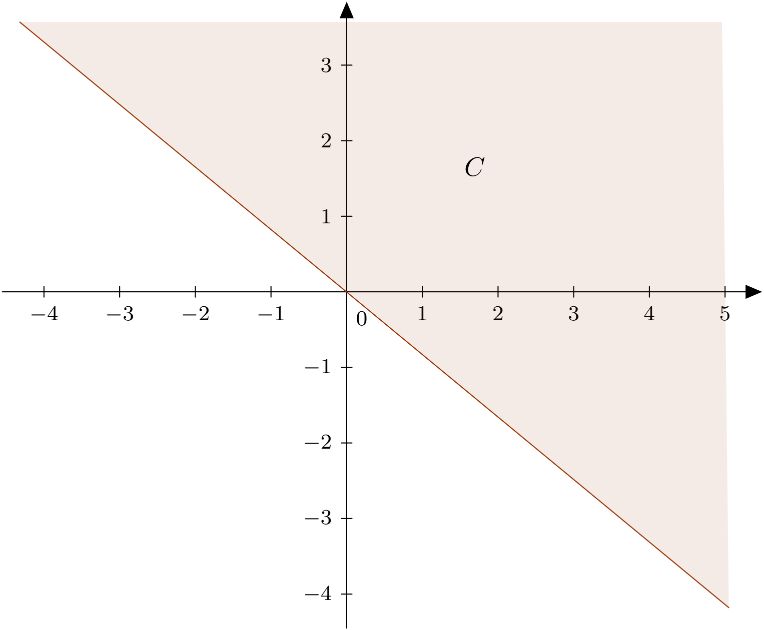
Convexity of C

Claim

The set C is convex.



[Back]

Holmström Characterization

- Set of allocations $A = \{a, b\}$
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- Consider $v_i(a) = v_i^*(a) + \epsilon$, $\epsilon > 0$, and write the DSIC constraint:

$$v_i^*(a) + \epsilon - p_{i,a} \geq v_i(b) - p_{i,b} \qquad (1)$$

outcome does not change $\Rightarrow$ payment does not change

Holmström Characterization

- Consider $v_i(a) = v_i^*(a) - \delta$, $\delta > 0$, and similarly:

$$v_i(b) - p_{i,b} \geq v_i^*(a) - \delta - p_{i,a} \tag{2}$$

- Combining Equations (1) and (2) and taking limits $\epsilon, \delta \to 0$, we get,

$$v_i(b) - p_{i,b} = v_i^*(a) - p_{i,a}$$

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- Substituting:

$$p_{i,a} - p_{i,b} = -\left(\sum_{j \in N \setminus \{i\}} v_j(b) - \sum_{j \in N \setminus \{i\}} v_j(a) \right)$$

[Back]